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A Uniqueness Theorem for Harmonic Functions

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Carlson's theorem [1, p. 153] states that an entire analytic function f of exponential type less than π (that is, one satisfying an inequality of the form $|f(z)| \leqslant Ae^{B|z|}$ with $B < \pi$) must vanish identically if it vanishes at the integers. At first sight there seems to be no possibility of extending Carlson's theorem to harmonic functions since an entire harmonic function ($\neq 0$) can have very slow growth and still vanish along a line (as $u(x, y) = y$ does), or even along two lines (as $u(x, y) = xy$ does). However, the possibility remains that a harmonic function of small exponential type might vanish identically if it vanishes at the lattice points, and this is in fact the case. We shall prove even more; it is enough to have it vanish on two parallel lines of lattice points.

THEOREM 1. *Let k be a positive integer and let $u(z)$ be a real-valued entire harmonic function of exponential type less than π/k. Let $u(m) = 0$ and $u(m + ik) = 0$ for $m = 0, \pm 1, \pm 2, \ldots$. Then $u(z) \equiv 0$.*

The type π/k is critical, since the imaginary part of $e^{\pi z/k}$ vanishes on the whole real axis and the whole line $y = k$.

For the proof of the theorem, let v be a harmonic function conjugate to u, so that $f(z) = u(z) + iv(z)$ is entire. It follows from Carathéodory's inequality [1, p. 2] that f is of exponential type less than π/k. In fact, if $A(r) = \max |u(z)|$ for $|z| = r$, Carathéodory's inequality states that

$$|f(z)| \leqslant |f(0)| + \frac{2r}{R - r}\{A(R) - u(0)\}, \qquad |z| = r < R.$$

If we take $R = r + 1$, we see that $f(z)$ is of the same exponential type as $u(z)$.

Now consider the entire function $f(z) + \overline{f(\bar{z})}$, which reduces to $2u(x)$ when

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$z = x$, and so vanishes at the integers. By Carlson's theorem, it vanishes identically, and so $f(z) = -\overline{f(\bar{z})}$; in particular,

$$f(z + ik) = -\overline{f(\bar{z} - ik)}. \tag{1}$$

Let $g(z) = f(z + ik)$. Then $g(z) + \overline{g(\bar{z})}$ also vanishes at the integers, so $g(z) = -\overline{g(\bar{z})}$, and this says that

$$f(z + ik) = -\overline{f(\bar{z} + ik)}.$$

Comparing this with (1), we get $f(z + ik) = f(z - ik)$; in other words, $f(z)$ has period $2ik$.

Now it is well known (see [2]) that an entire function $f(z)$ of exponential type and period $2ik$ must be an exponential sum of the form

$$f(z) = \sum_{j=-N}^{N} a_j e^{j\pi z/k}.$$

In the present case, we must have $N < 1$ since the type is less than π/k. That is, $f(z)$ is a constant; its real part u vanishes at the integers and so $u(z) \equiv 0$.

Theorem 1 suggests a number of problems, for example the following:

1. Is there a corresponding theorem for harmonic functions in three dimensions? That is, if a harmonic function u has sufficiently slow growth and $u(m, n, k) = 0$ for all lattice points (m, n, k), is it true that $u(x, y, z) \equiv 0$?

2. Is there a corresponding theorem for harmonic functions that vanish at equally spaced points on two intersecting lines? This is answered in Theorem 2, below.

3. Since a harmonic function of exponential type less than π is determined by its values on the lattice points m, $m + i$, it should be possible to reconstruct the function from these values. How?

4. Carlson's theorem has many extensions. Most of those dealing with functions that vanish at a sequence of points have obvious extensions to the present situation. However, it is not obvious whether we can extend, for example, Cartwright's theorem [1, p. 203] that an entire analytic function of exponential type less than π is bounded on the real axis if it is bounded at the integers.

THEOREM 2. *If u is a real-valued entire harmonic function of exponential type less than π, and $u(m) = u(me^{i\alpha}) = 0$, for $m = 0, \pm 1, \pm 2, \ldots$, then $u(z) \equiv 0$ unless α is a rational multiple of π.*

When α is a rational multiple p/q of π, the theorem fails, for example for $u(z) = r^q \sin q\theta$.

As in Theorem 1, construct $f(z) = u(z) + iv(z)$, and deduce again that $f(z) = -\overline{f(\bar{z})}$. Since $g(z) = f(ze^{i\alpha})$ satisfies the same conditions as $f(z)$, we have

$$f(ze^{i\alpha}) = -\overline{f(\bar{z}e^{i\alpha})} = -\overline{f(\overline{ze^{-i\alpha}})},$$

whence $f(ze^{2i\alpha}) = f(z)$. This is clearly impossible for a uniform $f(z)$ unless α is a rational multiple of π.

When $u(z)$ is the real part of an entire analytic function $f(z)$ that can be represented by the cardinal series

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{\sin \pi(z-n)}{\pi(z-n)} f(n),$$

there is an alternate line of proof for the conclusion of Theorem 1. Since $f(m) = ia_m$ is pure imaginary, demanding that $f(m+i)$ is pure imaginary leads to

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n a_n (m-n)}{(m-n)^2+1} = 0, \qquad m = 0, \pm 1, \pm 2, \ldots .$$

That is, the convolution of $\{a_n\}$ with

$$\left\{ \frac{(-1)^k k}{k^2+1} \right\}$$

is zero. Hence the product of the periodic functions F and G with these Fourier coefficients is zero. But $G(x) = (\pi/\sinh \pi) \sinh x$ on $(-\pi, \pi)$, and hence $F(z) = 0$ except at $x = 0$. Therefore all $a_n = 0$ and $f(z) \equiv 0$.

Added in proof. The proofs in this paper can be shortened by making use of results of Flatto, Newman and Shapiro [3].

References

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